This next remark allows us to make a sharp enough upper bound when we count the number of elements in $E_{span}(0, m, \frac{\beta}{2}, X \backslash U)$.

REMARK 4.10. *Suppose* $(y_0, \ldots, y_{l-1}) = \Phi_l(x)$ *for some* $x \in X$. *Then a point* $y_i \in E_{span}(0, m, \frac{\beta}{2}, X \backslash U)$ *can not be repeated in this* l *tuple.*

Proof: Suppose $y_j = y_k = z$ and $j < k$. Then we have $d_{(0,m)}([g,f]^{jm}(x), z) < \frac{\beta}{2}$ and $[g,f]^{jm}(x) \in X \backslash U$. Thus, $d_{(0,m)}([g,f]^{km}(x), z) < \frac{\beta}{2}$ and $[g,f]^{km}(x) \in X \backslash U$. This implies that $d([g,f]^{jm}(x), z) < \frac{\beta}{2}$ and $d([g,f]^{km}(x), z) < \frac{\beta}{2}$. Hence, $d([g,f]^{(k-j)m}([g,f]^{jm}(x)), [g,f]^{jm}(x)) = d([g,f]^{km}(x), [g,f]^{jm}(x)) < \beta$. By definition of β, this contradicts that $[g,f]^{jm}(x) \in X \backslash U$. ■

End of Remark 4.10.

Now choose $n \in \mathbb{N}$ so that $n > m \;\; r_{span}(0, m, \beta, \{f, g\}, X \backslash U)$. Suppose $E_{sep}(0, 2\epsilon, \{f, g\}, X)$ is a maximal $(0, n, 2\epsilon)$ separated set. For such an n, let l be the positive integer with $(l-1)m < n \leq lm$. The next Remark allows us to convert the problem of finding an upper bound for $E_{sep}(0, n, 2\epsilon, X)$, by finding an upper bound for $\Phi_l(E_{sep}(0, n, 2\epsilon, X))$.

REMARK 4.11. *The map* Φ_l *is* 1 *to* 1 *on* $E_{sep}(0, n, 2\epsilon, X)$.

Proof: Suppose $\Phi_l(x) = \Phi_l(z) = (y_0, \ldots, y_{l-1})$ where $x, z \in E_{sep}(0, n, 2\epsilon, X)$ For $0 \leq j < m$ and $0 \leq i < l$, we have

$d([g,f]^{im+j}(x), [g,f]^{im+j}(z)) = d([g,f]^{j}([g,f]^{im}(x)), [g,f]^{j}([g,f]^{im}(z)))$. Since m is a multiple of 2, $d([g,f]^{j}([g,f]^{im}(x)), [g,f]^{j}([g,f]^{im}(z))) \leq d([g,f]^{j}([g,f]^{im}(x)), y_i) +$

$d(y_i, [g,f]^j([g,f]^{im}(z)))$

$\leq d_{(0,m)}([g,f]^{im}(x), y_i) + d_{(0,m)}(y_i, [g,f]^{im}(z)) < \epsilon + \epsilon$ because of the definition of Φ_l and y_i.

The integer l is chosen so that $lm \geq n$. Since the above inequality holds for all i and j, then $d_{(0,n)}(x,z) < 2\epsilon$. Since the set $E_{sep}(0,n,2\epsilon,X)$ is $(0,n,2\epsilon)$ separated, then $x = z$. ■

End of Remark 4.11.

In this next Remark, we find an upper bound for the number of elements in $\Phi_l(E_{sep}(0,n,2\epsilon,X))$. Since q is fixed, notice that the spanning sets for $X\backslash U$ have a polynomial bound with degree q.

REMARK 4.12. *Set* $q = r_{span}(0,m,\beta,X\backslash U,\{f,g\})$ *and* $p = r_{span}(0,m,\epsilon,\Omega,\{f,g\})$ *Then* $|\Phi_l(E_{sep}(0,n,2\epsilon,X))| \leq (q+1)!l^q p^l$.

Proof: Let $\mathfrak{P}_j$ be the subset of l tuples in $\Phi_l(E_{sep}(0,n,2\epsilon,X))$ so that there are exactly j of the $y_i \in \{y_0,\ldots,y_{l-1}\}$ that are in $E_{span}(0,m,\beta,X\backslash U)$. Because the $y_i \in E_{span}(0,m,\beta,X\backslash U)$ can not be repeated in $\Phi_l(x)$ we have $j \leq r_{span}(0,m,\beta,X\backslash U,\{f,g\}) = q$. This bound, q, of j is independent of n and independent of l.

For $\mathfrak{P}_j$ there are $\binom{q}{j}$ ways of picking these j points $y_s \in E_{span}(0,m,\beta,X\backslash U)$. Further there are $l(l-1)\ldots(l-j+1) = \frac{l!}{(l-j)!}$ ways of arranging these choices in the ordered l tuples. Also, there are $r_{span}(0,m,\epsilon,\Omega,\{f,g\})^{l-j} = p^{l-j} \leq p^l$ ways of picking the remaining y_i from $E_{span}(0,m,\epsilon,\Omega)$. Hence, $|\mathfrak{P}_j| \leq \binom{q}{j}\frac{l!}{(l-j)!}p^l$. From this inequality, we have, $|\Phi_l(E_{sep}(0,n,2\epsilon,X))| = \sum_{j=0}^{q}|\mathfrak{P}_j| \leq \sum_{j=0}^{q}\binom{q}{j}\frac{l!}{(l-j)!}p^l$. By noting that $\binom{q}{j} \leq q!$ and $\frac{l!}{(l-j)!} \leq l^j \leq l^q$ we have $|\Phi^l(E_{sep}(0,n,2\epsilon,X))| \leq \sum_{j=0}^{q} q! l^q p^l$. Thus, $|\Phi_l(E_{sep}(0,n,2\epsilon,X))| \leq (q+1)! l^q p^l$. ■ End of Remark 4.12.

By Remarks 4.11 and 4.12, $r_{sep}(0,n,2\epsilon,X,\{f,g\}) = |\Phi_l(E_{sep}(0,n,2\epsilon,X))| \leq (q+1)! l^q p^l$ where $q = r_{span}(0,m,\beta,X \backslash U,\{f,g\})$ and $p = r_{span}(0,n,\epsilon,\Omega,\{f,g\})$.

Further, $h_{sep}(2\epsilon,\{f,g\},X) =$

$\limsup_{n\to\infty} \frac{1}{n} \log(r_{sep}(0,n,2\epsilon,X\{f,g\}))$

$\leq \limsup_{l\to\infty} \frac{1}{(l-1)m} \log[(q+1)! l^q p^l]$

$= \limsup_{l\to\infty} \frac{1}{(l-1)m} \{\log[(q+1)!] + q \log l + l \log p\}$

$\leq \frac{\log p}{m}$, since q is independent of l. Thus, $h_{sep}(2\epsilon,\{f,g\},X) \leq \frac{\log[r_{span}(0,m,\epsilon,\Omega,\{f,g\})]}{m}$.

Now note that we found an upper bound for when m is even, and further that for any $\eta > 0$ if m even is chosen large enough, we have

$\limsup_{m\to\infty} \frac{\log[r_{span}(0,m,\epsilon,\Omega,\{f,g\})]}{m} \leq \limsup_{m\to\infty} \frac{\log[r_{span}(0,m,\epsilon,\Omega,\{f,g\})]}{n} - \eta$.

Hence, $h_{sep}(2\epsilon,\{f,g\},X) \leq h_{span}(\epsilon,\{f,g\},\Omega) - \eta \leq h_{span}(\{f,g\},\Omega) - \eta$.

Since $\eta > 0$ was arbitrary, then $h_{sep}(2\epsilon,\{f,g\},X) \leq h_{span}(\{f,g\},\Omega)$.

Now let $\epsilon \to 0$, so $h_{sep}(\{f,g\},X) \leq h_{span}(\{f,g\},\Omega)$. From the other section, we learned that spanning and separating entropy are equivalent; thus, $h_{span}(\{f,g\},\Omega) = h_{sep}(\{f,g\},\Omega)$. This implies $h_{sep}(\{f,g\},X) \leq h_{sep}(\{f,g\},\Omega)$. Since $\Omega \subset X$, we obtain $h_{sep}(\{f,g\},X) \geq h_{sep}(\{f,g\},\Omega)$. ■ End of Theorem 4.13.

We stated earlier that we would present a proof of Remark 4.9 after the proof of the Theorem.

Proof: Suppose $y \in X \backslash U$. Then by the definition of Ω, there exists $\epsilon(y) > 0$ such that

$$(4.6) \qquad [g,f]^k(B(y,\epsilon(y))) \cap B(y,\epsilon(y)) = \emptyset$$

for all $k \geq 1$. Let $x \in B(y, \frac{\epsilon(y)}{2})$. If $p \in B(x, \frac{\epsilon(y)}{4})$ then $d(x,p) < \frac{\epsilon(y)}{4}$. Thus, $d(y,p) \leq d(y,x) + d(x,p) < 3\frac{\epsilon(y)}{4}$ which implies that $p \in B(y,\epsilon(y))$. Hence, $B(x, \frac{\epsilon(y)}{4}) \subset B(y,\epsilon(y))$ implies that $[g,f]^k(B(y, \frac{\epsilon(y)}{4})) \cap B(y, \frac{\epsilon(y)}{4}) = \emptyset$ for any $k \geq 1$. Hence, $\frac{\epsilon(y)}{4}$ will work for every $x \in B(y, \frac{\epsilon(y)}{2})$. Since $X \backslash U$ is compact, cover $X \backslash U$ with a finite number of $\frac{\epsilon(y)}{2}$ balls where $\epsilon(y)$ satisfies equation 4.6. Then we have $B(y_1, \frac{\epsilon(y_1)}{2}), \ldots, B(y_n, \frac{\epsilon(y_n)}{2})$ covering $X \backslash U$ and satisfying the discussion following equation 4.6. Set $\beta = \min\{\frac{\epsilon(y_1)}{4}, \ldots, \frac{\epsilon(y_n)}{4}\}$. Then $\beta > 0$ and from the previous discussion for any $x \in X \backslash U$, we obtain $[g,f]^k(B(x,\beta)) \cap B(x,\beta) = \emptyset$ for all $k \geq 1$.

■

THEOREM 4.14. *Suppose $(X, \{f_1, f_2, \ldots, f_r\})$ is a non-autonomous system with period r. The topological entropy of $(X, \{f_1, f_2, \ldots, f_r\})$ is equal to the topological entropy of $(\Omega, \{f_1, f_2, \ldots, f_r\})$ where Ω is the non-wandering set of $(X, \{f_1, f_2, \ldots, f_r\})$.*

Proof: As in the proof of Theorem 4.13 we choose a large m where $m = ra$ for some $a \in \mathbb{N}$. Then we proceed with the same steps as in Theorem 4.13. ■

We now turn our attention to exploring how changing the order in which functions are applied affects the entropy of the non-autonomous system. Throughout, we assume that our non-autonomous system is periodic. In particular, suppose $(X, \{f, g\})$ is a non-autonomous period 2 dynamical system. In this next part, we establish that $h(\{f,g\}) = h(\{g,f\})$. We first develop some Remarks and Lemmas which show that if we ignore whether the first iterate of a set is ϵ spanned or separated, the omission does not change the entropy. What affects the entropy is not the beginning of the sequence, but the tail of the sequence as in all limits.

REMARK 4.13. *If* $S \subset X$ *is* $(1, n, \epsilon, \{f, g\})$ *separated then* $f|_S$ *is* 1 *to* 1.

Proof: Suppose $x \neq y$ where $x, y \in S$. By contradiction, suppose $f(x) = f(y)$. Then $(g \circ f)^k(x) = (g \circ f)^k(y)$ for all $k \geq 1$. Thus, this means that $f \circ (g \circ f)^k(x) = f \circ (g \circ f)^k(y)$ for all $k \geq 1$. But this relationship implies that we can not $(1, n, \epsilon, \{f, g\})$ separate S. ■

LEMMA 4.6. *If for* $n \geq 2$, *we have* $S \subset X$ *is* $(1, n, \epsilon, \{f, g\})$ *separated, then* $T = f(S)$ *is* $(0, n-1, \epsilon, \{g, f\})$ *separated.*

Proof: Let $u, v \in T$ where $u \neq v$. By the definition of the image of a function, $T = f(S)$ implies there are points $x, y \in S$ with $x \neq y$ and $f(x) = u$ and $f(y) = v$. Since S is $(1, n, \epsilon, \{f, g\})$ separated there exists m with $1 \leq m < n$ so that one of the two following conditions holds:

$d((g \circ f)^k(x), (g \circ f)^k(y)) > \epsilon$ if $m = 2k$ where $k \geq 1$

OR

$d(f \circ (g \circ f)^k(x), f \circ (g \circ f)^k(y)) > \epsilon$ if $m = 2k + 1$ where $k \geq 0$.

The previous two conditions imply that

$d(g \circ (f \circ g)^j(f(x)), g \circ (f \circ g)^j(f(y))) > \epsilon$ where $j = k - 1$

OR

$d((f \circ g)^i(f(x)), (f \circ g)^i(f(y))) > \epsilon$ where $i = k$.

These two statements imply that

$d(g \circ (f \circ g)^j(u), g \circ (f \circ g)^j(v)) > \epsilon$ where $j = k - 1$

OR

$d((f \circ g)^i(u), (f \circ g)^i(v)) > \epsilon$ where $i = k$.

In the case where $m = 2k$ (where m is even), the assumption $k \geq 1$ implies that $2j + 1 = 2k - 2 + 1 \geq 0$, and $2j + 1 = 2(k - 1) = 2k - 1 = m - 1 < n - 1$. In the case where $m = 2k + 1$ (where m is odd) $k \geq 0$ implies that $2i \geq 0$, and $2i = 2k = m - 1 < n - 1$. Hence, T is $(0, n-1, \epsilon, \{g, f\})$ separated. ■

By Remark 4.6, $|T| = |f(S)|$ and from Lemma 4.6, we see that:

$$r_{sep}(1, n, \epsilon, \{f, g\}) \leq r_{sep}(0, n-1, \epsilon, \{g, f\}). \tag{4.7}$$

Using the same argument with the roles of f and g interchanged we see that

$$r_{sep}(1, n, \epsilon, \{g, f\}) \leq r_{sep}(0, n-1, \epsilon, \{f, g\}). \tag{4.8}$$

We find an upper bound for the maximal separating set, by splitting the separating set into the "tail" of the set, and the "head" of the set.

LEMMA 4.7. *Our upper bound estimate is* $r_{sep}(0, n, \epsilon, \{g, f\}) \leq r_{sep}(0, 1, \epsilon, \{g, f\}) + r_{sep}(1, n, \epsilon, \{g, f\})$

Proof: Suppose S is a $(0, n, \epsilon, \{g, f\})$ maximally separated set, i.e. $|S| = r_{sep}(0, n, \epsilon, \{g, f\})$. Set $\mathfrak{A} = \{A \subset S : \text{for every } a, b \in A \text{ with } a \neq b \text{ then } d(a, b) > \epsilon\}$. Since X is compact $\mathfrak{A}$ is a finite set, so it has a maximal element $A_m \in \mathfrak{A}$, where $|A_m| \geq |A|$ for any $A \in \mathfrak{A}$. Now, $|S| = |A_m| + |S \backslash A_m|$. Any two points $x, y \in S \backslash A_m$ where $x \neq y$ satisfy at least one of the two equations:

$d((f \circ g)^k(x), (f \circ g)^k(y)) > \epsilon$ where $k \geq 1$,

$d(g \circ (f \circ g)^k(x), g \circ (f \circ g)^k(y)) > \epsilon$ where $k \geq 0$.

Hence, $|S \setminus A_m| \leq r_{sep}(1, n, \epsilon, \{g, f\})$. Further, by the definition of A_m, $|A_m| \leq r_{sep}(0, 1, \epsilon, \{g, f\})$. Thus, $r_{sep}(0, n, \epsilon, \{g, f\}) \leq r_{sep}(0, 1, \epsilon, \{g, f\}) + r_{sep}(1, n, \epsilon, \{g, f\})$.
■

Now we establish that ignoring the first iterate does not change the value of the entropy.

LEMMA 4.8. *We have*

$$\limsup_{n\to\infty} \frac{\log r_{sep}(0, n, \epsilon, \{g, f\})}{n} = \limsup_{n\to\infty} \frac{\log r_{sep}(1, n, \epsilon, \{g, f\})}{n}.$$

Proof: First, note that $r_{sep}(0, 1, \epsilon, \{g, f\})$ is independent of n, so there exists a constant c where $r_{sep}(0, 1, \epsilon, \{g, f\}) \leq c$. Hence,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{\log[r_{sep}(0, 1, \epsilon, \{g, f\}) + r_{sep}(1, n, \epsilon, \{g, f\})]}{n} \\ &\leq \limsup_{n\to\infty} \frac{\log[c + r_{sep}(1, n, \epsilon, \{g, f\})]}{n} \\ &= \limsup_{n\to\infty} \frac{\log r_{sep}(1, n, \epsilon, \{g, f\})}{n}. \end{aligned}$$

Hence, by Lemma 4.7, and the previous inequality,

$$\limsup_{n\to\infty}\frac{\log r_{sep}(0,n,\epsilon,\{g,f\})}{n} \le \limsup_{n\to\infty}\frac{\log r_{sep}(1,n,\epsilon,\{g,f\})}{n}.$$

By the definition of r_{sep}, we see that $r_{sep}(1,n,\epsilon,\{g,f\}) \le r_{sep}(0,n,\epsilon,\{g,f\})$. Thus,

$$\limsup_{n\to\infty}\frac{\log r_{sep}(1,n,\epsilon,\{g,f\})}{n} \le \limsup_{n\to\infty}\frac{\log r_{sep}(0,n,\epsilon,\{g,f\})}{n}.$$

■

THEOREM 4.15. *Reversing the order of the function application of a period 2 non-autonomous system does not change the ϵ separated entropy,*

$$\limsup_{n\to\infty}\frac{\log r_{sep}(0,n,\epsilon,\{f,g\})}{n} = \limsup_{n\to\infty}\frac{\log r_{sep}(0,n,\epsilon,\{g,f\})}{n}.$$

This means that the topological entropy of $\{f,g,f,g,\dots\}$ is equal to the the topological entropy of $\{g,f,g,f,\dots\}$ i.e. $h(\{f,g\}) = h(\{g,f\})$.

Proof: By inequality 4.7, we have

$$\limsup_{n\to\infty}\frac{\log r_{sep}(1,n,\epsilon,\{f,g\})}{n}$$

$$\le \limsup_{n\to\infty}\frac{\log r_{sep}(0,n-1,\epsilon,\{g,f\})}{n}$$

$$\le \limsup_{n\to\infty}\frac{\log r_{sep}(0,n-1,\epsilon,\{g,f\})}{n-1}$$

$$= \limsup_{n\to\infty}\frac{\log r_{sep}(1,n-1,\epsilon,\{g,f\})}{n-1}.$$

Lemma 4.8 justifies the previous step. Thus, the first expression is

$$\leq \limsup_{n\to\infty} \frac{\log r_{sep}(1, n, \epsilon, \{g, f\})}{n-1}$$

by the definition of r_{sep}. The previous expression is

$$\leq \limsup_{n\to\infty} \frac{\log r_{sep}(0, n-1, \epsilon, \{f, g\})}{n-1}$$

because of inequality 4.8. The previous expression

$$= \limsup_{n\to\infty} \frac{\log r_{sep}(0, n, \epsilon, \{f, g\})}{n}$$

by the definition of lim sup.

By interchanging the roles of f and g and applying Lemma 4.8 we have

$$\limsup_{n\to\infty} \frac{\log r_{sep}(1, n, \epsilon, \{f, g\})}{n} = \limsup_{n\to\infty} \frac{\log r_{sep}(0, n, \epsilon, \{f, g\})}{n}.$$

Now we notice that we produced a chain of ordered inequalities and equalities where the first expression and last expression in the chain are equal. Hence,
$\limsup_{n\to\infty} \frac{\log rsep(1,n,\epsilon,\{f,g\})}{n} = \limsup_{n\to\infty} \frac{\log rsep(0,n-1,\epsilon,\{g,f\})}{n-1} = \limsup_{n\to\infty} \frac{\log rsep(0,n,\epsilon,\{f,g\})}{n}$. ■

Now that we have shown that $h(\{f, g\}) = h(\{g, f\})$, we use this fact about period 2 non-autonomous systems to prove an unintuitive fact about autonomous systems. In particular, we prove that $h(g \circ f) = \frac{1}{2}h(\{f, g\})$ and that $\frac{1}{2}h(\{g, f\}) = h(f \circ g)$; hence, $h(g \circ f) = h(f \circ g)$. In general, $g \circ f$ is a very different function than $f \circ g$. In general, these two functions are not topologically conjugate. We then extend this relationship to an arbitrary period r. This is important because it is an attempt to answer the question, to what extent does the behavior of a periodic non-autonomous dynamical system change, when we change the order in which the functions are applied? In particular, this question is interesting in the context of neural computation because one may not always train on a set of examples in the same order. It is important to discover when a trained neural network will behave differently by merely switching the order of a few examples presented during training.

REMARK 4.14. *If T is $(0, n, \epsilon, \{g \circ f\})$ separated, then T is $(0, 2n, \epsilon, \{f, g\})$ separated.*

Proof: Let $x, y \in T$. Then by definition $0 \leq k < n$ so that $d((g \circ f)^k(x), (g \circ f)^k(y)) > \epsilon$. This implies that $d([g, f]^{2k}(x), [g, f]^{2k}(y)] > \epsilon$, and $0 \leq 2k < 2n$. ∎

REMARK 4.15. *The non-autonomous system is an upper bound for the $g \circ f$ autonomous system, $r_{sep}(0, 2n, \epsilon, \{f, g\}) \geq r_{sep}(0, n, \epsilon, \{g \circ f\})$.*

Proof: Apply Remark 4.14 and the definition of r_{sep}.

LEMMA 4.9. *Our strict upper bound for epsilon entropy of $g \circ f$ is $h(\{f, g\}, \epsilon) \geq \frac{1}{2}h(\{g \circ f\}, \epsilon)$.*

Proof: We obtain the following chain of inequalities utilizing Remark 4.15. Thus, $h(\{f, g\}, \epsilon) = \limsup\limits_{k \to \infty} \frac{\log r_{sep}(0,k,\epsilon,\{f,g\})}{k}$

$$\geq \limsup_{n \to \infty} \frac{\log r_{sep}(0,2n,\epsilon,\{f,g\})}{2n}$$

$$\geq \limsup_{n \to \infty} \frac{\log r_{sep}(0,n,\epsilon,\{g \circ f\})}{2n} = \frac{1}{2} \limsup_{n \to \infty} \frac{\log r_{sep}(0,n,\epsilon,\{g \circ f\})}{n} = \frac{1}{2}h(g \circ f, \epsilon). \blacksquare$$

LEMMA 4.10. *The upper bound for the entropy of* $g \circ f$ *is* $h(\{f, g, \dots\}) \geq \frac{1}{2}h(g \circ f)$.

Proof: Apply Lemma 4.9 for ϵ approaching 0. ■

Now we work toward the inequality in the opposite direction, $h(\{f, g, \dots\}) \leq \frac{1}{2}h(g \circ f)$. We begin by proving a technical remark that relies on the uniform continuity of f and g. Since by assumption X is compact, f and g are uniformly continuous.

REMARK 4.16. *Let* $\epsilon > 0$. *There exists* $\delta_f(\epsilon) > 0$ *so that* $\delta_f(\epsilon) > 0$ $\delta_f(\epsilon) \leq \epsilon$ *and so that* $d(x, y) < \delta_f(\epsilon)$ *implies that* $d(f(x), f(y)) < \epsilon$. *Suppose* T *is a set that* $(0, n, \delta_f(\epsilon), g \circ f)$ *spans* X. *Then* T *also* $(0, 2n, \epsilon, \{f, g\})$ *spans* X.

Proof: Let $x \in X$. Then there exists $z \in T$ so that $d((g \circ f)^k(x), (g \circ f)^k(z) < \delta_f(\epsilon) \leq \epsilon$ where $0 \leq k < n$. To show that T $(0, 2n, \epsilon, \{f, g\})$ spans X the only part that remains are the functions $f \circ (g \circ f)^k$ where $0 \leq k < n$. But $d(f \circ (g \circ f)^k(x), f \circ (g \circ f)^k(z)) \leq \epsilon$ because $d((g \circ f)^k(x), (g \circ f)^k(z)) \leq \delta_f(\epsilon)$ ■

LEMMA 4.11. *The reverse inequality is* $h(\{f, g, \dots\}) \leq \frac{1}{2}h(g \circ f)$.

Proof: The value r_{span} represents the minimal number of elements that span X. Hence, by Remark 4.16, for any $\epsilon > 0$, we can find an $\eta = \delta_f(\epsilon) \leq \epsilon$. Thus, $r_{span}(0, 2n, \epsilon, \{f, g\})$
$\leq r_{span}(0, n, \delta_f(\epsilon), g \circ f)$. Thus, $h(\{f, g\}, \epsilon)$

$$= \limsup_{k \to \infty} \frac{\log r_{span}(0, k, \epsilon, \{f, g\})}{k}$$

$$\leq \limsup_{n\to\infty} \frac{\log r_{span}(0, 2n, \epsilon, \{f, g\})}{2n}$$

$$\leq \limsup_{n\to\infty} \frac{1}{2} \frac{\log r_{span}(0, n, \eta, g \circ f)}{n}$$

$$= \frac{1}{2} h(g \circ f, \eta).$$

Hence, for any $\epsilon > 0$ we were able to find an $\eta > 0$ so that $h(\{f, g\}, \epsilon) \leq h(g \circ f, \eta)$. Since $\epsilon_1 < \epsilon_2$ implies that $h(\{f_1, f_2, \dots\}, \epsilon_2) \leq h(\{f_1, f_2, \dots\}, \epsilon_1)$, we obtain $h(\{f, g\}) \leq \frac{1}{2} h(g \circ f)$. ■

THEOREM 4.16. *If X is a compact metric space, and $f, g : X \longrightarrow X$ are continuous functions, then $h(\{f, g\}) = \frac{1}{2} h(g \circ f)$.* ■

Proof: The result is an immediate consequence of Lemma 4.11 and Lemma 4.10. ■

COROLLARY 4.5. *Period 2 function composition commutes with respect to topological entropy, $h(g \circ f) = h(f \circ g)$.*

Proof: By Theorem 4.15, $h(\{f, g\}) = h(\{g, f\})$. ■

Now, by using Corollary 4.5 and induction, we extend this result to period n.